

# PLURISUBHARMONIC FUNCTIONS WITH WEAK SINGULARITIES

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*Dedicated to Professor C.O. Kiselman  
on the occasion of his retirement*

ABSTRACT. We study the complex Monge-Ampère operator in bounded hyperconvex domains of  $\mathbb{C}^n$ . We introduce several classes of weakly singular plurisubharmonic functions : these are functions of finite weighted Monge-Ampère energy. They generalize the classes introduced by U.Cegrell, and give a stratification of the space of (almost) all unbounded plurisubharmonic functions. We give an interpretation of these classes in terms of the speed of decreasing of the Monge-Ampère capacity of sublevel sets and solve associated complex Monge-Ampère equations.

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## 1. INTRODUCTION

In two seminal papers [Ce 1,2], U.Cegrell was able to define and study the complex Monge-Ampère operator  $(dd^c \cdot)^n$  on special classes of unbounded plurisubharmonic functions in a hyperconvex domain in  $\mathbb{C}^n$ .

Since we are considering a new and important scale of classes of plurisubharmonic functions with finite weighted Monge-Ampère energy, we find it convenient to introduce new notations which reflect our intuition. Therefore we have to modify some of the classical ones to avoid confusions.

Let  $\Omega \subset \mathbb{C}^n$  be a bounded hyperconvex domain. The first important class considered by Cegrell (denoted by  $\mathcal{E}_0(\Omega)$  in [Ce1]), is the class  $\mathcal{T}(\Omega)$  of plurisubharmonic “test functions” on  $\Omega$ , i.e. the convex cone of all bounded plurisubharmonic functions  $\varphi$  defined on  $\Omega$  such that  $\lim_{z \rightarrow \zeta} \varphi(z) = 0$ , for every  $\zeta \in \partial\Omega$ , and  $\int_{\Omega} (dd^c \varphi)^n < +\infty$ . Besides this class, we will need the following classes introduced in [Ce1], [Ce2].

- The class  $DMA(\Omega)$  is the set of plurisubharmonic functions  $u$  such that for all  $z_0 \in \Omega$ , there exists a neighborhood  $V_{z_0}$  of  $z_0$  and  $u_j \in \mathcal{T}(\Omega)$  a decreasing sequence which converges towards  $u$  in  $V_{z_0}$  and satisfies  $\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty$ . U.Cegrell has shown [Ce 2] that the operator  $(dd^c \cdot)^n$  is well defined on  $DMA(\Omega)$  and continuous under decreasing limits. The class  $DMA(\Omega)$  is stable under taking maximum and it is the largest class with these properties (Theorem 4.5 in [Ce 2]). Actually this class, introduced and denoted by  $\mathcal{E}(\Omega)$  by U.Cegrell ([Ce 2]), turns out to coincide with the domain of definition of the complex Monge-Ampère operator on  $\Omega$  as was shown by Z.Blocki [Bl 1,2];

- the class  $\mathcal{F}(\Omega)$  is the “global version” of  $DMA(\Omega)$ : a function  $u$  belongs to  $\mathcal{F}(\Omega)$  iff there exists  $u_j \in \mathcal{T}(\Omega)$  a sequence decreasing towards  $u$  in all of  $\Omega$ , which satisfies  $\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty$ ;
- the class  $\mathcal{F}_a(\Omega)$  is the set of functions  $u \in \mathcal{F}(\Omega)$  whose Monge-Ampère measure  $(dd^c u)^n$  is absolutely continuous with respect to capacity i.e. it does not charge pluripolar sets;
- the class  $\mathcal{E}^p(\Omega)$  (respectively  $\mathcal{F}^p(\Omega)$ ) is the set of functions  $u$  for which there exists a sequence of functions  $u_j \in \mathcal{T}(\Omega)$  decreasing towards  $u$  in all of  $\Omega$ , and so that  $\sup_j \int_{\Omega} (-u_j)^p (dd^c u_j)^n < +\infty$  (respectively  $\sup_j \int_{\Omega} [1 + (-u_j)^p] (dd^c u_j)^n < +\infty$ ).

One purpose of this article is to use the formalism developed in [GZ] in a compact setting to give a unified treatment of all these classes. Given an increasing function  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ , we consider the set  $\mathcal{E}_{\chi}(\Omega)$  of plurisubharmonic functions of finite  $\chi$ -weighted Monge-Ampère energy. These are functions  $u \in PSH(\Omega)$  such that there exists  $u_j \in \mathcal{T}(\Omega)$  decreasing to  $u$ , with

$$\sup_{j \in \mathbb{N}} \int_{\Omega} (-\chi) \circ u_j (dd^c u_j)^n < +\infty.$$

It will be shown that  $\mathcal{E}_{\chi}(\Omega) \subset DMA(\Omega)$ .

Many important properties follow from the elementary observation that the Monge-Ampère measures  $1_{\{u > -j\}} (dd^c u_j)^n$  strongly converge towards  $(dd^c u)^n$  in the set  $\Omega \setminus \{u = -\infty\}$ , when  $u_j := \max(u, -j)$  are the “canonical approximants” of  $u$ :

**Theorem A.** *If  $u \in DMA(\Omega)$ , then for all Borel sets  $B \subset \Omega \setminus \{u = -\infty\}$ ,*

$$\int_B (dd^c u)^n = \lim_{j \rightarrow \infty} \int_{B \cap \{u > -j\}} (dd^c u_j)^n,$$

where  $u_j := \max(u, -j)$  are the canonical approximants.

We establish this result in *section 2* and derive several consequences. This yields in particular simple proofs of quite general comparison principles.

The classes  $\mathcal{E}_{\chi}(\Omega)$  have very different properties, depending on whether  $\chi(0) = 0$  or  $\chi(0) \neq 0$ ,  $\chi(-\infty) = -\infty$  or  $\chi(-\infty) \neq -\infty$ ,  $\chi$  is convex or concave. We study these in *section 3* and give a capacity interpretation of them in *section 4*. Let us stress in particular Corollary 4.3 which gives an interesting characterization of the class  $\mathcal{E}^p(\Omega)$  of U.Cegrell, in terms of the speed of decreasing of the capacity of sublevel sets:

**Proposition B.** *For any real number  $p > 0$ ,*

$$\mathcal{E}^p(\Omega) = \left\{ \varphi \in PSH^-(\Omega); \int_0^{+\infty} (-\varphi)^{n+p-1} Cap_{\Omega}(\{\varphi < -t\}) dt < +\infty \right\}.$$

Here  $Cap_{\Omega}$  denotes the Monge-Ampère capacity introduced by E. Bedford and B.A. Taylor ([BT1]). Of course  $\mathcal{E}^p(\Omega) = \mathcal{E}_{\chi}(\Omega)$ , for  $\chi(t) := -(-t)^p$ .

Our formalism allows us to consider further natural subclasses of  $PSH(\Omega)$ , especially functions with finite “high-energy” (when  $\chi$  increases faster than polynomials at infinity). We study in *section 5* the range of the Monge-Ampère operator on these classes. Given a positive finite Borel measure  $\mu$

on  $\Omega$ , we set

$$F_\mu(t) := \sup\{\mu(K); K \subset \Omega \text{ compact}, \text{Cap}_\Omega(K) \leq t\}, t \geq 0.$$

Observe that  $F := F_\mu$  is an increasing function on  $\mathbb{R}^+$  which satisfies

$$\mu(K) \leq F(\text{Cap}_\Omega(K)), \quad \text{for all Borel subsets } K \subset X.$$

The measure  $\mu$  does not charge pluripolar sets iff  $F(0) = 0$ .

When  $F(x) \lesssim x^\alpha$  vanishes at order  $\alpha > 1$ , S. Kolodziej has proved [K 2] that the equation  $\mu = (dd^c \varphi)^n$  admits a unique *continuous* solution with  $\varphi|_{\partial\Omega} = 0$ . If  $F(x) \lesssim x^\alpha$  with  $0 < \alpha < 1$ , it follows from the work of U. Cegrell [Ce 1] that there is a unique solution in some class  $\mathcal{F}^p(\Omega)$ .

Another objective of this article is to fill in the gap inbetween Cegrell's and Kolodziej's results, by considering all intermediate dominating functions  $F$ . Write  $F(x) = x[\varepsilon(-\ln x/n)]^n$  where  $\varepsilon : \mathbb{R}^+ \rightarrow [0, \infty[$  is nonincreasing.

Our second main result is:

**Theorem C.** *Assume for all compact subsets  $K \subset \Omega$ ,*

$$\mu(K) \leq F_\varepsilon(\text{Cap}_\Omega(K)), \text{ where } F_\varepsilon(x) = x[\varepsilon(-\ln x/n)]^n.$$

*Then there exists a unique function  $\varphi \in \mathcal{F}(\Omega)$  such that  $\mu = (dd^c \varphi)^n$  and*

$$\text{Cap}_\Omega(\{\varphi < -s\}) \leq \exp(-nH^{-1}(s)), \text{ for all } s > 0,$$

*Here  $H^{-1}$  is the reciprocal function of  $H(x) = e \int_0^x \varepsilon(t)dt + s_0(\mu)$ .*

*In particular  $\varphi \in \mathcal{E}_\chi(\Omega)$  where  $-\chi(-t) = \exp(nH^{-1}(t)/2)$ .*

Note in particular that when  $\mu \leq \text{Cap}_\Omega$  (i.e.  $\varepsilon \equiv 1$ ), then  $\mu = (dd^c \varphi)^n$  for a function  $\varphi \in \mathcal{F}(\Omega)$  such that  $\text{Cap}_\Omega(\{\varphi < -s\})$  decreases exponentially fast. Simple examples show that this bound is sharp (see [BGZ]).

For similar results in the case of compact Kähler manifolds, we refer the reader to [GZ], [EGZ], [BGZ].

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## 2. CANONICAL APPROXIMANTS

We let  $PSH(\Omega)$  denote the set of plurisubharmonic functions on  $\Omega$  (psh for short), and fix  $u \in PSH(\Omega)$ . E. Bedford and B.A. Taylor have defined in [BT 2] the non pluripolar part of the Monge-Ampère measure of  $u$ : the sequence  $\mu_u^{(j)} := 1_{\{u > -j\}}(dd^c \max[u, -j])^n$  is a nondecreasing sequence of positive measures. Its limit  $\mu_u$  is the “nonpluripolar part of  $(dd^c u)^n$ ”, defined as,

$$\mu_u(B) = \lim_{j \rightarrow \infty} \int_{B \cap \{u > -j\}} (dd^c \max[u, -j])^n,$$

for any Borel set  $B \subset \Omega$ .

In general  $\mu_u$  is not locally bounded near  $\{u = -\infty\}$  (see e.g. [Ki]), but if  $u \in DMA(\Omega)$  then  $\mu_u$  is a regular Borel measure:

**Theorem 2.1.** *If  $u \in DMA(\Omega)$ , then for all Borel sets  $B \subset \Omega \setminus \{u = -\infty\}$ ,*

$$\int_B (dd^c u)^n = \lim_{j \rightarrow \infty} \int_{B \cap \{u > -j\}} (dd^c u_j)^n,$$

where  $u_j := \max(u, -j)$ . In particular,  $\mu_u = 1_{\{u > -\infty\}}(dd^c u)^n$ .

The measure  $(dd^c u)^n$  puts no mass on pluripolar sets  $E \subset \{u > -\infty\}$ .

*Proof.* Note that this convergence result is local in nature, hence we can assume, without loss of generality, that  $u \in \mathcal{F}(\Omega)$ . For  $s > 0$  consider the psh function  $h_s := \max(u/s + 1, 0)$ . Observe that  $h_s$  increases to the Borel function  $1_{\{u > -\infty\}}$  and  $\{h_s = 0\} = \{u \leq -s\}$ . We claim that

$$h_s(dd^c \max(u, -s))^n = h_s(dd^c u)^n, \text{ for all } s > 0,$$

in the sense of measures on  $\Omega$ .

Indeed, recall that we can find a sequence of *continuous* tests functions  $u_k$  in  $\mathcal{T}(\Omega)$  decreasing towards  $u$  (see Theorem 2.1 in [Ce 2]). It follows from Proposition 5.1 in [Ce 2] that  $h_s(dd^c \max(u_k, -s))^n$  converges weakly to  $h_s(dd^c \max(u, -s))^n$  and  $h_s(dd^c u_k)^n$  converges weakly to  $h_s(dd^c u)^n$  as  $k \rightarrow \infty$ .

Since  $\max(u_k, -s) = u_k$  on  $\{u_k > -s\}$ , which is an open neighborhood of the set  $\{u > -s\}$ , we infer

$$h_s(dd^c \max(u, -s))^n = h_s(dd^c u)^n,$$

as claimed.

Observe that

$$h_s(dd^c \max(u, -s))^n = h_s 1_{\{u > -s\}}(dd^c u)^n = h_s \mu_u^{(s)}$$

increases as  $s \uparrow +\infty$  towards  $1_{\{u > -\infty\}} \mu_u = \mu_u$ , as follows from the monotone convergence and Radon-Nikodym theorems. Similarly  $h_s(dd^c u)^n$  converges to  $1_{\{u > -\infty\}}(dd^c u)^n$ . Thus  $\mu_u = 1_{\{u > -\infty\}}(dd^c u)^n$ , this shows the desired convergence on any Borel set  $B \subset \Omega \setminus \{u = -\infty\}$ .  $\square$

Note that if  $u \in \mathcal{F}_a(\Omega)$  then  $\int_B (dd^c u)^n = \lim_{j \rightarrow \infty} \int_B (dd^c u_j)^n$ , for all Borel subsets  $B \subset \Omega$  (see Theorem 3.4).

As an application, we give a simple proof of the following general version of the comparison principle (see also [NP]).

**Theorem 2.2.** *Let  $u \in DMA(\Omega)$  and  $v \in PSH^-(\Omega)$ . Then*

$$1_{\{u > v\}}(dd^c u)^n = 1_{\{u > v\}}(dd^c \max(u, v))^n$$

*Proof.* Set  $u_j = \max(u, -j)$  and  $v_j = \max(v, -j)$ . Recall from [BT 2] that the desired equality is known for bounded psh functions,

$$1_{\{u_j > v_{j+1}\}}(dd^c u_j)^n = 1_{\{u_j > v_{j+1}\}}(dd^c \max(u_j, v_{j+1}))^n.$$

Observe that  $\{u > v\} \subset \{u_j > v_{j+1}\}$ , hence

$$\begin{aligned} 1_{\{u > v\}} \cdot 1_{\{u > -j\}}(dd^c u_j)^n &= 1_{\{u > v\}} \cdot 1_{\{u > -j\}}(dd^c \max(u, v, -j))^n \\ &= 1_{\{u > v\}} \cdot 1_{\{\max(u, v) > -j\}}(dd^c \max(u, v, -j))^n. \end{aligned}$$

It follows from Theorem 2.1 that  $1_{\{u > -j\}}(dd^c u_j)^n$  converges in the strong sense of Borel measures towards  $\mu_u = 1_{\{u > -\infty\}}(dd^c u)^n$ . Observe that  $1_{\{u > v\}} 1_{\{u > -\infty\}} = 1_{\{u > v\}}$ . We infer, by using Theorem 2.1 again with  $\max(u, v)$ , that

$$1_{\{u > v\}}(dd^c u)^n = 1_{\{u > v\}}(dd^c \max(u, v))^n.$$

$\square$

The following result has been proved by U.Cegrell [Ce 3]. We provide here a simple proof using Theorem 2.2, yet another consequence of the fact that the Monge-Ampère measures  $1_{\{u > -j\}}(dd^c u_j)^n$  strongly converge towards  $1_{\{u > -\infty\}}(dd^c u)^n$  when  $u_j := \max(u, -j)$  are the “canonical approximants” (Theorem 2.1).

**Corollary 2.3.** *Let  $\varphi \in \mathcal{F}(\Omega)$  and  $u \in DMA(\Omega)$  such that  $u \leq 0$ . Then*

$$\int_{\{\varphi < u\}} (dd^c u)^n \leq \int_{\{\varphi < u\} \cup \{\varphi = -\infty\}} (dd^c \varphi)^n$$

*Proof.* Since  $\psi := \max\{u, \varphi\} \in \mathcal{F}(\Omega)$  and  $\varphi \leq \psi$  on  $\Omega$ , it follows that

$$\int_{\Omega} (dd^c \psi)^n \leq \int_{\Omega} (dd^c \varphi)^n.$$

Indeed this is clear when  $\varphi \in \mathcal{T}(\Omega)$  by integration by parts and follows by approximation when  $\varphi \in \mathcal{F}(\Omega)$  (see [Ce 2]).

We infer by using Theorem 2.2,

$$\begin{aligned} \int_{\{\varphi < u\}} (dd^c u)^n &= \int_{\{\varphi < u\}} (dd^c \max(u, \varphi))^n \\ &= \int_{\Omega} (dd^c \max(u, \varphi))^n - \int_{\{\varphi \geq u\}} (dd^c \max(u, \varphi))^n \\ &\leq \int_{\Omega} (dd^c \varphi)^n - \int_{\{\varphi > u\}} (dd^c \varphi)^n - \int_{\{\varphi = u\}} (dd^c \max(u, \varphi))^n \\ &\leq \int_{\{\varphi \leq u\}} (dd^c \varphi)^n \end{aligned}$$

Now take  $0 < \varepsilon < 1$  and apply the previous result to get

$$\int_{\{\varepsilon \varphi < u\}} (dd^c u)^n \leq \int_{\{\varepsilon \varphi \leq u\}} (dd^c \varepsilon \varphi)^n = \varepsilon^n \int_{\{\varepsilon \varphi \leq u\}} (dd^c \varphi)^n.$$

The desired inequality follows by letting  $\varepsilon \rightarrow 1$ , since  $\{\varepsilon \varphi < u\}$  increases to  $\{\varphi < u\}$  and  $\{\varepsilon \varphi \leq u\}$  increases to  $\{\varphi < u\} \cup \{\varphi = -\infty\}$ .  $\square$

Note that Corollary 2.3 is still valid when  $\varphi, u \in DMA(\Omega)$  under the condition  $\{\varphi < u\} \Subset \Omega$ .

The following comparison principle is due to U.Cegrell (see Theorem 5.15 in [Ce 2] and Theorem 3.7 in [Ce 3]).

**Corollary 2.4.** *Let  $\varphi \in \mathcal{F}_a(\Omega)$  and  $u \in DMA(\Omega)$ , such that  $(dd^c \varphi)^n \leq (dd^c u)^n$ . Then  $u \leq \varphi$ .*

*In particular if  $(dd^c u)^n = (dd^c \varphi)^n$  with  $u, \varphi \in \mathcal{F}_a(\Omega)$ , then  $u = \varphi$ .*

*Proof.* The proof is a consequence of Corollary 2.3 and follows from standard arguments (see e.g. [BT 1] for bounded psh function).  $\square$

Note that the result still holds when  $u \in DMA(\Omega)$  is such that  $(dd^c u)^n$  vanishes on pluripolar sets and  $u \geq v$  near  $\partial\Omega$ . However it fails in  $\mathcal{F}(\Omega)$  (see [Ce 2] and [Z]).

Now, as another consequence of Theorem 2.2, we provide the following result which will be useful in the sequel:

**Corollary 2.5.** *Fix  $\varphi \in \mathcal{F}(\Omega)$ . Then for all  $s > 0$  and  $t > 0$ ,*

$$(2.1) \quad t^n \text{Cap}_\Omega(\{\varphi < -s - t\}) \leq \int_{(\varphi < -s)} (dd^c \varphi)^n \leq s^n \text{Cap}_\Omega(\{\varphi < -s\}).$$

*In particular*

$$(2.2) \quad \int_\Omega (dd^c \varphi)^n = \lim_{s \downarrow 0} s^n \text{Cap}_\Omega(\varphi \leq -s) = \sup_{s > 0} s^n \text{Cap}_\Omega(\varphi < -s).$$

*Moreover a negative function  $u \in \text{PSH}(\Omega)$  belongs to  $\mathcal{F}(\Omega)$  if and only if  $\sup_{s > 0} s^n \text{Cap}_\Omega(u < -s) < +\infty$*

The inequalities (2.1) was proved for psh test functions in [K3] (see also [CKZ] and [EGZ]). For  $\varphi \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ , it follows by approximation and quasi-continuity. In the general case, it can be deduced using Theorem 2.1. The last assertion follows easily from (2.1). It was first obtained in ([B]).

### 3. WEIGHTED ENERGY CLASSES

**Definition 3.1.** *Let  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  be an increasing function. We let  $\mathcal{E}_\chi(\Omega)$  denote the set of all functions  $u \in \text{PSH}(\Omega)$  for which there exists a sequence  $u_j \in \mathcal{T}(\Omega)$  decreasing to  $u$  in  $\Omega$  and satisfying*

$$\sup_{j \in \mathbb{N}} \int_\Omega (-\chi) \circ u_j (dd^c u_j)^n < \infty.$$

This definition clearly contains the classes of U.Cegrell:

- $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$  if  $\chi$  is bounded and  $\chi(0) \neq 0$ ;
- $\mathcal{E}_\chi(\Omega) = \mathcal{E}^p(\Omega)$  if  $\chi(t) = -(-t)^p$ ;
- $\mathcal{E}_\chi(\Omega) = \mathcal{F}^p(\Omega)$  if  $\chi(t) = -1 - (-t)^p$ .

We will give hereafter interpretation of the classes  $\mathcal{F}(\Omega) \cap L^\infty(\Omega)$  and  $\mathcal{F}_a(\Omega)$  in terms of weighted-energy as well.

Let us stress that the classes  $\mathcal{E}_\chi(\Omega)$  are very different whether  $\chi(0) \neq 0$  (finite total Monge-Ampère mass) or  $\chi(0) = 0$ .

To simplify we consider in this section the case  $\chi(0) \neq 0$ , so that all functions under consideration have a well defined Monge-Ampère measure of finite total mass in  $\Omega$ . Note however that many results to follow still hold when  $\chi(0) = 0$ .

**Proposition 3.2.** *Let  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  be an increasing function such that  $\chi(-\infty) = -\infty$  and  $\chi(0) \neq 0$ . Then*

$$\mathcal{E}_\chi(\Omega) \subset \mathcal{F}_a(\Omega).$$

*In particular the Monge-Ampère measure  $(dd^c u)^n$  of a function  $u \in \mathcal{E}_\chi(\Omega)$  is well defined and does not charge pluripolar sets. More precisely,*

$$\mathcal{E}_\chi(\Omega) = \{u \in \mathcal{F}(\Omega) / \chi \circ u \in L^1((dd^c u)^n)\}.$$

*Proof.* Fix  $u \in \mathcal{E}_\chi(\Omega)$  and  $u_j \in \mathcal{T}(\Omega)$  a defining sequence such that

$$\sup_j \int_\Omega \chi(u_j) (dd^c u_j)^n < +\infty.$$

The condition  $\chi(0) \neq 0$  implies that  $\mathcal{E}_\chi(\Omega) \subset \mathcal{F}(\Omega)$ . In particular the Monge-Ampère measure  $(dd^c u)^n$  is well defined. It follows from the upper semi-continuity of  $u$  that  $-\chi(u)(dd^c u)^n$  is bounded from above by any

cluster point of the bounded sequence  $-\chi(u_j)(dd^c u_j)^n$ . Therefore  $\int_{\Omega} (-\chi) \circ u (dd^c u)^n < +\infty$ , in particular  $(dd^c u)^n$  does not charge the set  $\{\chi(u) = -\infty\}$ , which coincides with  $\{u = -\infty\}$ , since  $\chi(-\infty) = -\infty$ . It follows therefore from Theorem 2.1 that the measure  $(dd^c u)^n$  does not charge pluripolar sets.

To prove the last assertion, it remains to show the reverse inclusion

$$\mathcal{E}_{\chi}(\Omega) \supset \{u \in \mathcal{F}(\Omega) / \chi \circ u \in L^1((dd^c u)^n)\}.$$

So fix  $u \in \mathcal{F}(\Omega)$  such that  $\chi \circ u \in L^1((dd^c u)^n)$ . It follows from [K 1] that there exists, for each  $j \in \mathbb{N}$ , a function  $u_j \in \mathcal{T}(\Omega)$  such that  $(dd^c u_j)^n = \mathbf{1}_{\{u > j\rho\}}(dd^c u)^n$ , where  $\rho \in \mathcal{T}(\Omega)$  any defining function for  $\Omega = \{\rho < 0\}$ . Observe that  $(dd^c u)^n \geq (dd^c u_{j+1})^n \geq (dd^c u_j)^n$ . We infer from Corollary 2.4 that  $(u_j)$  is a decreasing sequence and  $u \leq u_j$ . The monotone convergence theorem thus yields

$$\int_{\Omega} (-\chi) \circ u_j (dd^c u_j)^n = \int_{\Omega} (-\chi) \circ u_j \mathbf{1}_{\{u > j\rho\}} (dd^c u)^n \rightarrow \int_{\Omega} (-\chi) \circ u (dd^c u)^n < +\infty,$$

so that  $u \in \mathcal{E}_{\chi}(\Omega)$ .  $\square$

There is a natural partial ordering of the classes  $\mathcal{E}_{\chi}(\Omega)$  : if  $\chi = O(\tilde{\chi})$  then  $\mathcal{E}_{\tilde{\chi}}(\Omega) \subset \mathcal{E}_{\chi}(\Omega)$ . Classes  $\mathcal{E}_{\chi}(\Omega)$  provide a full scale of subclasses of  $PSH^-(\Omega)$  of unbounded functions, reaching, “at the limit”, bounded plurisubharmonic functions.

**Proposition 3.3.**

$$\mathcal{F}(\Omega) \cap L^{\infty}(\Omega) = \bigcap_{\substack{\chi(0) \neq 0 \\ \chi(-\infty) = -\infty}} \mathcal{E}_{\chi}(\Omega),$$

where the intersection runs over all increasing functions  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ .

Note that it suffices to consider here those functions  $\chi$  which are *concave*.

*Proof.* One inclusion is clear. Namely if  $u \in \mathcal{F}(\Omega) \cap L^{\infty}(\Omega)$  and  $u_j \in \mathcal{T}(\Omega)$  are decreasing to  $u$ , then for any  $\chi$  as above,

$$\int_{\Omega} -\chi(u_j)(dd^c u_j)^n \leq \left[ \sup_{\Omega} |\chi(u)| \right] \int_{\Omega} (dd^c u)^n < +\infty.$$

Conversely, assume  $u \in \mathcal{F}(\Omega)$  is unbounded. Then the sublevel sets  $\{u < t\}$  are non empty for all  $t < 0$ , hence we can consider the function  $\chi$  such that

$$t \mapsto \chi'(t) = \frac{1}{(dd^c u)^n(\{u < t\})}, \text{ for all } t < 0.$$

The function  $\chi$  is clearly increasing. Moreover  $(dd^c u)^n$  has finite (positive) mass, hence  $\chi'(t) \geq \frac{1}{(dd^c u)^n(\Omega)}$ . This yields  $\chi(-\infty) = -\infty$ . Now

$$\int_{\Omega} (-\chi) \circ u (dd^c u)^n = \int_0^{+\infty} \chi'(-s)(dd^c u)^n(\{u < -s\})ds = +\infty.$$

This shows that if  $u \in \mathcal{E}_{\chi}(\Omega)$  for all  $\chi$  as above, then  $u$  has to be bounded.  $\square$

When  $u \in \mathcal{E}_{\chi}(\Omega) \subset \mathcal{F}_a(\Omega)$ , the canonical approximants  $u_j := \max(u, -j)$  yield strong convergence properties of weighted Monge-Ampère operators:

**Theorem 3.4.** *Let  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  be an increasing function such that  $\chi(-\infty) = -\infty$  and  $\chi(0) \neq 0$ . Fix  $u \in \mathcal{E}_\chi(\Omega)$  as set  $u^j = \max(u, -j)$ . Then for each Borel subset  $B \subset \Omega$ ,*

$$\lim_{j \rightarrow +\infty} \int_B \chi(u^j)(dd^c u^j)^n = \int_B \chi(u)(dd^c u)^n.$$

*Moreover if  $(u_j)_{j \in \mathbb{N}}$  is any decreasing sequence in  $\mathcal{E}_\chi(\Omega)$  converging to  $u$  such that  $\sup_j \int_\Omega |\chi(u_j)|(dd^c u_j)^n < +\infty$ , then*

$$\lim_{j \rightarrow +\infty} \int_\Omega \chi(u_j)(dd^c u_j)^n = \int_\Omega \chi(u)(dd^c u)^n.$$

Let us stress that this convergence result is stronger than Theorem 5.6 in [Ce 1]: on one hand we produce here an explicit (and canonical) sequence of bounded approximants, on the other hand the convergence holds in the strong sense of Borel measures. Moreover the  $\chi$ -energy is continuous under decreasing sequences of plurisubharmonic functions with uniformly bounded  $\chi$ -energies.

*Proof.* We first show that  $(dd^c u^j)^n$  converges towards  $(dd^c u)^n$  “in the strong sense of Borel measures”, i.e.  $(dd^c u^j)^n(B) \rightarrow (dd^c u)^n(B)$ , for any Borel set  $B \subset \Omega$ . Observe that for  $j \in \mathbb{N}^*$  fixed and  $0 < s < j$ ,  $\{u < -s\} = \{u_j < -s\}$ . It follows from Corollary 2.5 that

$$\int_\Omega (dd^c u^j)^n = \int_\Omega (dd^c u)^n.$$

Therefore

$$\begin{aligned} \int_{\{u \leq -j\}} (dd^c u^j)^n &= \int_\Omega (dd^c u^j)^n - \int_{\{u > -j\}} (dd^c u^j)^n \\ &= \int_\Omega (dd^c u)^n - \int_{\{u > -j\}} (dd^c u)^n = \int_{\{u \leq -j\}} (dd^c u)^n. \end{aligned}$$

Thus if  $B \subset \Omega$  is a Borel subset,

$$\begin{aligned} \left| \int_B (dd^c u^j)^n - \int_B (dd^c u)^n \right| &\leq \int_{\{u \leq -j\}} (dd^c u^j)^n + \int_{\{u \leq -j\}} (dd^c u)^n \\ &\leq 2 \int_{\{u \leq -j\}} (dd^c u)^n \rightarrow 0, \text{ as } j \rightarrow +\infty. \end{aligned}$$

The proof that  $\chi \circ u^j (dd^c u^j)^n$  converges strongly towards  $\chi \circ u (dd^c u)^n$  goes along similar lines, once we observe that

$$\begin{aligned} \int_{\{u \leq -j\}} -\chi \circ u^j (dd^c u^j)^n &= -\chi(-j) \int_{\{u \leq -j\}} (dd^c u^j)^n = \\ &= -\chi(-j) \int_{\{u \leq -j\}} (dd^c u)^n \leq \int_{\{u \leq -j\}} -\chi \circ u (dd^c u)^n. \end{aligned}$$

To prove the second statment we proceed as in [GZ]. Observe that the statement is true for uniformly bounded sequences of plurisubharmonic functions by Bedford and Taylor convergence theorems. For the general case, we first consider an increasing function  $\tilde{\chi} : \mathbb{R}^- \rightarrow \mathbb{R}^-$  such that  $\tilde{\chi} = o(\chi)$



and prove the convergence of the  $\tilde{\chi}$ -energies. Indeed, for  $k \in \mathbb{N}$  define the canonical approximants

$$u_j^k := \sup\{u_j, -k\}, \quad \text{and} \quad u^k := \sup\{u, -k\}.$$

The integer  $k$  being fixed, the sequence  $(u_j^k)_{j \in \mathbb{N}}$  is uniformly bounded and decreases towards  $u^k$ , hence the  $\tilde{\chi}$ -energies of  $u_j^k$  converge to the  $\tilde{\chi}$ -energy of  $u^k$  as  $j \rightarrow +\infty$ . Thus we will be done if we can show that the  $\tilde{\chi}$ -energies of  $u_j^k$  converge to the  $\tilde{\chi}$ -energy of  $u_j$  uniformly in  $j$  as  $k \rightarrow +\infty$ . This follows easily from the following inequalities

$$\begin{aligned} I(j, k) &:= \left| \int_{\Omega} \tilde{\chi}(u_j^k) (dd^c u_j^k)^n - \int_{\Omega} \tilde{\chi}(u_j) (dd^c u_j)^n \right| \\ &\leq \int_{\{u_j \leq -k\}} -\tilde{\chi}(u_j^k) (dd^c u_j^k)^n + \int_{\{u_j \leq -k\}} -\tilde{\chi}(u_j) (dd^c u_j)^n \\ &\leq \frac{\tilde{\chi}(-k)}{\chi(-k)} \left( \int_{\{u_j \leq -k\}} -\chi(u_j^k) (dd^c u_j)^n + \int_{\{u_j \leq -k\}} -\chi(u_j) (dd^c u_j)^n \right) \\ &\leq 2 \frac{\tilde{\chi}(-k)}{\chi(-k)} \int_{\Omega} -\chi(u_j) (dd^c u_j)^n \leq 2M \frac{\tilde{\chi}(-k)}{\chi(-k)}, \end{aligned}$$

where  $M := \sup_j \int_{\Omega} -\chi(u_j) (dd^c u_j)^n < +\infty$  and the last inequality follows from previous computations.

For the general case, observe that  $0 \leq f := -\chi(u) \in L^1((dd^c u)^n)$  by Proposition 3.2. Then it follows easily by an elementary integration theory argument that there exists an increasing function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\lim_{t \rightarrow +\infty} h(t)/t = +\infty$  and  $h(f) \in L^1((dd^c u)^n)$  (see [RR]). Thus  $u \in \mathcal{E}_{\chi_1}(\Omega)$ , where  $\chi_1(t) := -h(-\chi(t))$  for  $t < 0$  and  $\chi = o(\chi_1)$  and the continuity property for  $\chi$ -energies follows from the previous case.  $\square$

#### 4. CAPACITY ESTIMATES

Of particular interest for us here are the classes  $\mathcal{E}_{\chi}(\Omega)$ , where the weight  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  has fast growth at infinity. It is useful in practice to understand these classes through the speed of decreasing of the capacity of sublevel sets.

The Monge-Ampère capacity has been introduced and studied by E. Bedford and A. Taylor in [BT 1]. Given  $K \subset \Omega$  a Borel subset, it is defined as

$$\text{Cap}_{\Omega}(K) := \sup \left\{ \int_K (dd^c u)^n / u \in PSH(\Omega), -1 \leq u \leq 0 \right\}.$$

**Definition 4.1.**

$$\hat{\mathcal{E}}_{\chi}(\Omega) := \left\{ \varphi \in PSH(\Omega) / \int_0^{+\infty} t^n \chi'(-t) \text{Cap}_{\Omega}(\{\varphi < -t\}) dt < +\infty \right\}.$$

The classes  $\mathcal{E}_{\chi}(\Omega)$  and  $\hat{\mathcal{E}}_{\chi}(\Omega)$  are closely related:

**Proposition 4.2.** *The classes  $\hat{\mathcal{E}}_{\chi}(\Omega)$  are convex and stable under maximum: if  $\varphi \in \hat{\mathcal{E}}_{\chi}(\Omega)$  and  $\psi \in PSH^-(\Omega)$ , then  $\max(\varphi, \psi) \in \hat{\mathcal{E}}_{\chi}(\Omega)$ .*

One always has  $\hat{\mathcal{E}}_\chi(\Omega) \subset \mathcal{E}_\chi(\Omega)$ , while

$$\mathcal{E}_{\hat{\chi}}(\Omega) \subset \hat{\mathcal{E}}_\chi(\Omega), \text{ where } \hat{\chi}(t) = \chi(2t).$$

*Proof.* The convexity of  $\hat{\mathcal{E}}_\chi(\Omega)$  follows from the following simple observation: if  $\varphi, \psi \in \hat{\mathcal{E}}_\chi(\Omega)$  and  $0 \leq a \leq 1$ , then

$$\{a\varphi + (1-a)\psi < -t\} \subset \{\varphi < -t\} \cup \{\psi < -t\}.$$

The stability under maximum is obvious.

Assume  $\varphi \in \hat{\mathcal{E}}_\chi(\Omega)$ . We can assume without loss of generality  $\varphi \leq 0$  and  $\chi(0) = 0$ . Set  $\varphi_j := \max(\varphi, -j)$ . It follows from Corollary 2.5 that

$$\begin{aligned} \int_{\Omega} (-\chi) \circ \varphi_j (dd^c \varphi_j)^n &= \int_0^{+\infty} \chi'(-t) (dd^c \varphi_j)^n (\varphi_j < -t) dt \\ &\leq \int_0^{+\infty} \chi'(-t) t^n \text{Cap}_{\Omega}(\varphi < -t) dt < +\infty, \end{aligned}$$

This shows that  $\varphi \in \mathcal{E}_\chi(\Omega)$ . The other inclusion goes similarly, using the second inequality in Corollary 2.5

Observe that  $\mathcal{E}_{\hat{\chi}}(\Omega) \subset \hat{\mathcal{E}}_\chi(\Omega)$ , with  $\hat{\chi}(t) = \chi(2t)$ , as follows by applying inequalities of Corollary 2.5 with  $t = s$ . □

Observe that  $\mathcal{E}_{\hat{\chi}}(\Omega) = \mathcal{E}_\chi(\Omega)$  when  $\chi(t) = -(-t)^p$ . We thus obtain a characterization of U.Cegrell's classes  $\mathcal{E}^p(\Omega)$  in terms of the speed of decreasing of the capacity of sublevel sets. This is quite useful since this second definition does not use the Monge-Ampère measure of the function (nor of its approximants):

**Corollary 4.3.**

$$\mathcal{E}^p(\Omega) = \left\{ \varphi \in PSH^-(\Omega) / \int_0^{+\infty} t^{n+p-1} \text{Cap}_{\Omega}(\{\varphi < -t\}) dt < +\infty \right\}.$$

This also provide us with a characterization of the class  $\mathcal{F}_a(\Omega)$ :

**Corollary 4.4.**

$$\mathcal{F}_a(\Omega) = \bigcup_{\substack{\chi(0) \neq 0, \\ \chi(-\infty) = -\infty}} \mathcal{E}_\chi(\Omega).$$

As we shall see in the proof, it is sufficient to consider here functions  $\chi$  that are *convex*.

*Proof.* The inclusion  $\supset$  follows from Proposition 3.2. To prove the reverse inclusion, it suffices to show that if  $u \in \mathcal{F}_a(\Omega)$  then there exists a function  $\chi$  such that  $u \in \hat{\mathcal{E}}_\chi(\Omega)$ : this is because  $\cup \mathcal{E}_\chi = \cup \hat{\mathcal{E}}_\chi$ . Set

$$h(t) := t^n \text{Cap}_{\Omega}(\{u < -t\}) \text{ and } \tilde{h}(t) := \sup_{s > t} h(s), \quad t > 0$$

The function  $\tilde{h}$  is bounded, decreasing and converges to zero at infinity. Consider  $\chi(t) := \frac{-1}{\sqrt{\tilde{h}(-t)}}$  for all  $t < 0$ . Thus  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  is convex increasing,

with  $\chi(0) \neq 0$  and  $\chi(-\infty) = -\infty$ . Moreover

$$\int_0^{+\infty} t^n \chi'(-t) \text{Cap}_\Omega(\{\varphi < -t\}) dt \leq \frac{1}{2} \int_0^{+\infty} \frac{-\tilde{h}'(s)}{\tilde{h}^{1/2}(s)} ds = \tilde{h}^{1/2}(0) < +\infty,$$

as follows from Corollary 2.5.  $\square$

Let us observe that a negative psh function  $u$  belongs to  $\mathcal{F}(\Omega)$  if and only if  $\tilde{h}(0) < +\infty$  (see Corollary 2.5).

We end up this section with the following useful observation. Let  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  be a non-constant concave increasing function. Its inverse function  $\chi^{-1} : \mathbb{R}^- \rightarrow \mathbb{R}^-$  is convex, hence for all  $\varphi \in PSH(\Omega)$ , the function  $\chi^{-1} \circ \varphi$  is plurisubharmonic,

$$dd^c \chi^{-1} \circ \varphi = (\chi^{-1})' \circ \varphi dd^c \varphi + (\chi^{-1})'' d\varphi \wedge d^c \varphi \geq 0.$$

Now

$$\text{Cap}_\Omega(\{\chi^{-1} \circ \varphi < -t\}) = \text{Cap}_\Omega(\{\varphi < \chi(-t)\})$$

decreases (very) fast if  $\chi$  has (very) fast growth at infinity. Thus  $\chi^{-1} \circ \varphi$  belongs to some class  $\mathcal{E}_{\hat{\chi}}(\Omega)$ , where  $\hat{\chi}$  is completely determined by  $\chi$  and has approximately the same growth order. This shows in particular that the class  $\mathcal{E}_\chi(\Omega)$  characterizes pluripolar sets, whatever the growth of  $\chi$ :

**Theorem 4.5.** *Let  $P \subset \Omega$  be a (locally) pluripolar set. Then for any concave increasing function  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  with  $\chi(-\infty) = -\infty$ , there exists  $\varphi \in \mathcal{E}_\chi(\Omega)$  such that*

$$P \subset \{\varphi = -\infty\}.$$

In particular we can choose  $\varphi \in \mathcal{E}_{\exp}(\Omega)$ , where

$$\mathcal{E}_{\exp}(\Omega) := \left\{ \varphi \in \mathcal{F}(\Omega); \int_\Omega e^{-\varphi} (dd^c \varphi)^n < +\infty \right\}.$$

## 5. THE RANGE OF THE COMPLEX MONGE-AMPÈRE OPERATOR

Throughout this section,  $\mu$  denotes a fixed positive Borel measure of finite total mass  $\mu(\Omega) < +\infty$  which is dominated by the Monge-Ampère capacity. We want to solve the following Monge-Ampère equation

$$(dd^c \varphi)^n = \mu, \quad \text{with } \varphi \in \mathcal{F}(\Omega),$$

and measure how far the (unique) solution  $\varphi$  is from being bounded, by assuming that  $\mu$  is suitable dominated by the Monge-Ampère capacity.

Measures dominated by the Monge-Ampère capacity have been extensively studied by S.Kolodziej in [K 1,2,3]. The main result of his study, achieved in [K 2], can be formulated as follows. Fix  $\varepsilon : \mathbb{R} \rightarrow [0, \infty[$  a continuous decreasing function and set  $F_\varepsilon(x) := x[\varepsilon(-\ln x/n)]^n$ . If for all compact subsets  $K \subset \Omega$ ,

$$\mu(K) \leq F_\varepsilon(\text{Cap}_\Omega(K)), \quad \text{and} \quad \int_0^{+\infty} \varepsilon(t) dt < +\infty,$$

then  $\mu = (dd^c \varphi)^n$  for some continuous function  $\varphi \in PSH(\Omega)$  with  $\varphi|_{\partial\Omega} = 0$ .

The condition  $\int_0^{+\infty} \varepsilon(t) dt < +\infty$  means that  $\varepsilon$  decreases fast enough towards zero at infinity. This gives a quantitative estimate on how fast  $\varepsilon(-\ln \text{Cap}_\Omega(K)/n)$ , hence  $\mu(K)$ , decreases towards zero as  $\text{Cap}_\Omega(K) \rightarrow 0$ .

When  $\int^{+\infty} \varepsilon(t)dt = +\infty$ , it is still possible to show that  $\mu = (dd^c\varphi)^n$  for some function  $\varphi \in \mathcal{F}(\Omega)$ , but  $\varphi$  will generally be unbounded. We now measure how far it is from being so:

**Theorem 5.1.** *Assume for all compact subsets  $K \subset \Omega$ ,*

$$(5.1) \quad \mu(K) \leq F_\varepsilon(\text{Cap}_\Omega(K)).$$

*Then there exists a unique function  $\varphi \in \mathcal{F}(\Omega)$  such that  $\mu = (dd^c\varphi)^n$ , and*

$$\text{Cap}_\Omega(\{\varphi < -s\}) \leq \exp(-nH^{-1}(s)), \text{ for all } s > 0,$$

*Here  $H^{-1}$  is the reciprocal function of  $H(x) = e \int_0^x \varepsilon(t)dt + e\varepsilon(0) + \mu(\Omega)^{1/n}$ .*

*In particular  $\varphi \in \mathcal{E}_\chi(\Omega)$  with  $-\chi(-t) = \exp(nH^{-1}(t)/2)$ .*

For examples showing that these estimates are essentially sharp, we refer the reader to section 4 in [BGZ].

*Proof.* The assumption on  $\mu$  implies in particular that it vanishes on pluripolar sets. It follows from [Ce 2] that there exists a unique  $\varphi \in \mathcal{F}_a(\Omega)$  such that  $(dd^c\varphi)^n = \mu$ . Set

$$f(s) := -\frac{1}{n} \log \text{Cap}_\Omega(\{\varphi < -s\}), \quad \forall s > 0.$$

The function  $f$  is increasing and  $f(+\infty) = +\infty$ , since  $\text{Cap}_\Omega$  vanishes on pluripolar sets.

It follows from Corollary 2.5 and (5.1) that for all  $s > 0$  and  $t > 0$ ,

$$t^n \text{Cap}_\Omega(\varphi < -s - t) \leq \mu(\varphi < -s) \leq F_\varepsilon(\text{Cap}_\Omega(\{\varphi < -s\})).$$

Therefore

$$(5.2) \quad \log t - \log \varepsilon \circ f(s) + f(s) \leq f(s + t).$$

We define an increasing sequence  $(s_j)_{j \in \mathbb{N}}$  by induction. Setting

$$s_{j+1} = s_j + e\varepsilon \circ f(s_j), \text{ for all } j \in \mathbb{N}.$$

*The choice of  $s_0$ .* We choose  $s_0 \geq 0$  large enough so that  $f(s_0) \geq 0$ . We must insure that  $s_0 = s_0(\mu)$  can be chosen to be independent of  $\varphi$ . It follows from Corollary 2.5 that

$$\text{Cap}_\Omega(\{\varphi < -s\}) \leq \frac{\mu(\Omega)}{s^n}, \quad \forall s > 0$$

hence  $f(s) \geq \log s - 1/n \log \mu(\Omega)$ . Therefore  $f(s_0) \geq 0$  if  $s_0 = \mu(\Omega)^{1/n}$ .

*The growth of  $s_j$ .* We can now apply (5.2) and get  $f(s_j) \geq j + f(s_0) \geq j$ . Thus  $\lim_j f(s_j) = +\infty$ . There are two cases to be considered.

If  $s_\infty = \lim s_j \in \mathbb{R}^+$ , then  $f(s) \equiv +\infty$  for  $s > s_\infty$ , i.e.  $\text{Cap}_\Omega(\varphi < -s) = 0$ ,  $\forall s > s_\infty$ . Therefore  $\varphi$  is bounded from below by  $-s_\infty$ , in particular  $\varphi \in \mathcal{E}_\chi(\Omega)$  for all  $\chi$ .

Assume now ( second case) that  $s_j \rightarrow +\infty$ . For each  $s > 0$ , there exists  $N = N_s \in \mathbb{N}$  such that  $s_N \leq s < s_{N+1}$ . We can estimate  $s \mapsto N_s$ ,

$$\begin{aligned} s \leq s_{N+1} &= \sum_0^N (s_{j+1} - s_j) + s_0 = \sum_0^N e \varepsilon \circ f(s_j) + s_0 \\ &\leq e \sum_0^N \varepsilon(j) + s_0 \leq e \int_0^N \varepsilon(t) dt + \tilde{s}_0 =: H(N), \end{aligned}$$

where  $\tilde{s}_0 = s_0 + e \varepsilon(0)$ . Therefore  $H^{-1}(s) \leq N \leq f(s_N) \leq f(s)$ , hence

$$Cap_\Omega(\varphi < -s) \leq \exp(-nH^{-1}(s)).$$

Set now  $g(t) = -\chi(-t) = \exp(nH^{-1}(t)/2)$ . Then

$$\begin{aligned} \int_0^{+\infty} t^n g'(t) Cap_\Omega(\varphi < -t) dt \\ \leq \frac{n}{2} \int_0^{+\infty} t^n \frac{1}{\varepsilon(H^{-1}(t)) + s_0} \exp(-nH^{-1}(t)/2) dt \\ \leq C \int_0^{+\infty} (t+1)^n \exp(n(\alpha-1)t) dt < +\infty. \end{aligned}$$

This shows that  $\varphi \in \mathcal{E}_\chi(\Omega)$  where  $\chi(t) = -\exp(nH^{-1}(-t)/2)$ .  $\square$

Observe that the proof above gives easily an a priori uniform bound of the solution of  $(dd^c \varphi)^n = \mu$ , when  $\mu$  is a finite Borel mesure on  $\Omega$  satisfying (5.1) with  $\int_0^{+\infty} \varepsilon(t) dt < +\infty$  (see also [K2]). Indeed it follows from the above estimates that  $\varphi \geq -s_\infty$ , where

$$s_\infty \leq e \int_0^{+\infty} \varepsilon(t) dt + e \varepsilon(0) + \mu(\Omega)^{1/n}.$$

We now generalize U.Cegrell's main result [Ce 1].

**Theorem 5.2.** *Let  $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$  be an increasing function such that  $\chi(-\infty) = -\infty$ . Suppose there exists a locally bounded function  $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\limsup_{t \rightarrow +\infty} F(t)/t < 1$ , and*

$$(5.3) \quad \int_\Omega (-\chi) \circ u \, d\mu \leq F(E_\chi(u)), \quad \forall u \in \mathcal{T}(\Omega),$$

where  $E_\chi(u) := \int_\Omega (-\chi) \circ u (dd^c u)^n$  denotes the  $\chi$ -energy of  $u$ .

Then there exists a function  $\varphi \in \mathcal{E}_\chi(\Omega)$  such that  $\mu = (dd^c \varphi)^n$ .

*Proof.* The assumption on  $\mu$  implies in particular that it vanishes on pluripolar sets. It follows from [Ce 2] that there exists a function  $u \in \mathcal{T}(\Omega)$  and  $f \in L_{loc}^1((dd^c u)^n)$  such that  $\mu = f(dd^c u)^n$ .

Consider  $\mu_j := \min(f, j)(dd^c u)^n$ . This is a finite measure which is bounded from above by the Monge-Ampère measure of a bounded function. It follows therefore from [K 1] that there exist  $\varphi_j \in \mathcal{T}(\Omega)$  such that

$$(dd^c \varphi_j)^n = \min(f, j)(dd^c u)^n.$$

The comparison principle shows that  $\varphi_j$  is a decreasing sequence. Set  $\varphi = \lim_{j \rightarrow \infty} \varphi_j$ . It follows from (5.3) that  $E_\chi(\varphi_j)(F(E_\chi(\varphi_j)))^{-1} \leq 1$ , hence  $\sup_{j \geq 1} E_\chi(\varphi_j) < \infty$ . This yields  $\varphi \in \mathcal{E}_\chi(\Omega)$ .

We conclude now by continuity of the Monge-Ampère operator along decreasing sequences that  $(dd^c\varphi)^n = \mu$ .  $\square$

When  $\chi(t) = -(-t)^p$  (class  $\mathcal{F}^p(\Omega)$ ),  $p \geq 1$ , the above result was established by U.Cegrell in [Ce 1]. Condition (5.3) is also necessary in this case, and the function  $F$  can be made quite explicit: there exists  $\varphi \in \mathcal{F}^p(\Omega)$  such that  $\mu = (dd^c\varphi)^n$  if and only if  $\mu$  satisfies (5.3) with  $F(t) = Ct^{p/(p+n)}$ , for some constant  $C > 0$ .

Actually the measure  $\mu$  satisfies (5.3) for  $\chi(t) = -(-t)^p$ , and  $F(t) = C \cdot t^{p/(p+n)}$ ,  $p > 0$  if and only if  $\mathcal{F}^p(\Omega) \subset L^p(\mu)$  (see [GZ]).

We finally remark that this condition can be interpreted in terms of domination by capacity.

**Proposition 5.3.** *If  $\mathcal{F}^p(\Omega) \subset L^p(\mu)$ , then there exists  $C > 0$  such that*

$$\mu(K) \leq C \cdot \text{Cap}_\Omega(K)^{\frac{p}{p+n}}, \text{ for all } K \subset \Omega.$$

*Conversely if  $\mu(\cdot) \lesssim \text{Cap}_\Omega^\alpha(\cdot)$  for some  $\alpha > p/(p+n)$ , then  $\mathcal{F}^p(\Omega) \subset L^p(\mu)$ .*

*Proof.* The estimate (5.3) applied to  $u = u_K^*$ , the relative extremal function of the compact  $K$ , yields

$$\begin{aligned} \mu(K) &= \int_\Omega 1_K \cdot d\mu \leq \int_\Omega (-u_K^*)^p d\mu \\ &\leq C \cdot \left( \int_\Omega (-u_K^*)^p (dd^c u_K^*)^n \right)^{\frac{p}{p+n}} \\ &= C \cdot [\text{Cap}_\Omega(K)]^{\frac{p}{n+p}}. \end{aligned}$$

Conversely, assume that  $\mu(K) \leq C \cdot \text{Cap}_\Omega^\alpha(K)$  for all compact  $K \subset \Omega$ , where  $\alpha > p/(n+p)$  then (5.3) is satisfied. Indeed, if  $u \in \mathcal{F}^p(\Omega)$ , then

$$\begin{aligned} \int_\Omega (-u)^p d\mu &= p \int_1^\infty t^{p-1} \mu(u < -t) dt + O(1) \\ &\leq C \cdot p \int_1^\infty t^{p-1} (\text{Cap}_\Omega(u < -t))^\alpha dt + O(1) \\ &\leq C \cdot \left( \int_1^\infty t^{n+p-1} \text{Cap}_\Omega(u < -t) dt \right)^\alpha \cdot \left( \int_1^\infty t^{[p-1-\alpha(n+p-1)]/\beta} dt \right)^\beta + O(1), \end{aligned}$$

where  $\alpha + \beta = 1$ . The first integral converges by Corollary 4.3, the latter one is finite since  $p-1-\alpha(n+p-1) > \alpha-1 = -\beta$ .  $\square$

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